The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 ***Version 2.0***

Date: November 7, 2023 Course: EE 313 Evans

Name: ____
Last, First

- This in-person exam is scheduled to last 75 minutes.
- Open books, open notes, and open class materials, including homework assignments and solution sets and previous midterm exams and solutions.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please mute all computer systems.
- Please turn off all phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation.

Problem	*Point Value*	*Your score*	*Topic*
1	27		System Properties
2	24		Convolution
3	27		System Identification
4	22		Filter Design
Total	100		

SPFirst Sec. 5-4, 5-5, 8-2 & 8-4.2	HW 5.2
Lecture Slides 8-3 to 8-6, 8-8 & 12-11 to 12-15	
F18 Midterm Prob 2.5 & F21 Midterm Prob 2.1	

Problem 2.1. *System Properties*. 27 points.

Each discrete-time system has input $x[n]$ and output $y[n]$, and $x[n]$ and $y[n]$ might be complex-valued.

Determine if each system is linear or nonlinear, time-invariant or time-varying, and bounded-input bounded-output (BIBO) stable or unstable.

You must either prove that the system property holds in the case of linearity, time-invariance, or stability, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Part	*System Name*	*System Formula*	*Linear?*	*Time-Invariant?*	*BIBO Stable?*
(a)	First-Order Difference Filter	$y[n] = x[n] - x[n-1]$ for $n \geq 0$ and $x[-1] = 0$	**Yes**	**Yes**	**Yes**
(b)	Amplitude Modulation	$y[n] = x[n] \cos(\hat{\omega}_0\, n)$ for $n \geq 0$ where $\hat{\omega}_0$ is a constant	**Yes**	**No**	**Yes**
(c)	Exponentiation	$y[n] = e^{x[n]}$ for $-\infty < n < \infty$	**No**	**Yes**	**Yes**

***Linearity*. We'll first apply the all-zero input test. If the output is not zero for all time, then the system is not linear. Otherwise, we'll have to apply the definitions for homogeneity and additivity. All-zero input test is a special case of homogeneity $\boldsymbol{a\, x[n] \rightarrow a\, y[n]}$ when the constant $\boldsymbol{a = 0}$.**

***Stability*. Bounded input $\boldsymbol{|x[n]| \leq B < \infty}$ would give bounded output $\boldsymbol{|y[n]| \leq C < \infty}$.**

(a) First-Order Difference Filter: $y[n] = x[n] - x[n-1]$ for $n \geq 0$ and $x[-1] = 0$. *9 points*. *HW 5.2*

***Linearity*: Passes all-zero input test. Initial condition is zero, necessary for LTI to hold.**

- ***Homogeneity*: Input $\boldsymbol{a\, x[n]}$. Output is** $\boldsymbol{y_{scaled}[n] = (a\, x[n]) - (a\, x[n])_{n \rightarrow n-1} = a\, x[n] - a\, x[n-1] = a\, y[n]}$**. YES.**
- ***Additivity*. Input $\boldsymbol{x_1[n] + x_2[n]}$. Output is** $\boldsymbol{y_{additive}[n] = (x_1[n] + x_2[n]) - (x_1[n] + x_2[n])_{n \rightarrow n-1} = (x_1[n] + x_2[n]) - (x_1[n-1] + x_2[n-1]) = x_1[n] - x_1[n-1] + x_2[n] - x_2[n-1] = y_1[n] - y_2[n]}$**. YES.**

***T-I*: Input $\boldsymbol{x[n-n_0]}$. Output $\boldsymbol{y_{shifted}[n] = x[n-n_0] + x[n-n_0-1] = y[n-n_0]}$. YES.**

Stability: $\boldsymbol{|y[n]| = |x[n] - x[n-1]| \leq |x[n]| + |x[n-1]| = B + B = 2B}$**. YES.**

(b) Amplitude Modulation: $y[n] = x[n] \cos(\hat{\omega}_0\, n)$ for $n \geq 0$ where $\hat{\omega}_0$ is a constant. *9 points*. *HW 5.2*

***Linearity*: Passes all-zero input test. No initial conditions.**

- ***Homogeneity*: Input $\boldsymbol{a\, x[n]}$. Output $\boldsymbol{y_{scaled}[n] = (a\, x[n]) \cos(\hat{\omega}_0\, n) = a\, y[n]}$. YES.**
- ***Additivity*. Input $\boldsymbol{x_1[n] + x_2[n]}$. Output is**
  $\boldsymbol{y_{additive}[n] = (x_1[n] + x_2[n]) \cos(\hat{\omega}_0\, n)}$
  $\boldsymbol{y_{additive}[n] = x_1[n] \cos(\hat{\omega}_0\, n) + x_2[n] \cos(\hat{\omega}_0\, n) = y_1[n] + y_2[n]}$**. YES.**

***T-I*: Input $\boldsymbol{x[n-n_0]}$. Output $\boldsymbol{y_{shifted}[n] = x[n-n_0] \cos(\hat{\omega}_0\, n) \neq y[n-n_0]}$. NO.**

Stability: $\boldsymbol{|y[n]| = |x[n] \cos(\hat{\omega}_0\, n)| \leq |x[n]|\, |\cos(\hat{\omega}_0\, n)| \leq B}$ **because** $\boldsymbol{|\cos(\hat{\omega}_0\, n)| \leq 1}$**. YES.**

(c) Exponentiation: $y[n] = e^{x[n]}$ for $-\infty < n < \infty$. *9 points*.

***Linearity*: Does not pass the all-zero input test; i.e., when $\boldsymbol{x[n] = 0}$, $\boldsymbol{y[n] = e^0 = 1 \neq 0}$. NO.**

***T-I*: Input $\boldsymbol{x[n-n_0]}$. Output $\boldsymbol{y_{shifted}[n] = e^{x[n-n_0]} = y[n-n_0]}$. Pointwise systems are T-I. YES.**

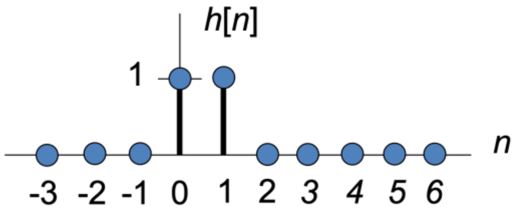
Stability: $\boldsymbol{|y[n]| = \left|e^{x[n]}\right| = \left|e^{x_{real}[n] + j\, x_{imag}[n]}\right| = \left|e^{x_{real}[n]}\right|\left|e^{j\, x_{imag}[n]}\right| \leq \left|e^{x_{real}[n]}\right| \leq e^{|x_{real}[n]|} \leq 1 + e^B}$ **because $\boldsymbol{e^v}$ is a non-negative monotonic function of real variable $\boldsymbol{v}$. YES.**

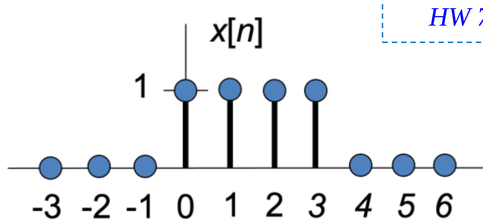
SPFirst Sec. 5-6, 8-2 | Handout E Convolution
F17 Midterm 2.1(b) & 2.2(b) and F21 Midterm 2.2
Lecture 8 | HW 5.1, 6.1 & 7.1 | Tune-Ups 5 & 6

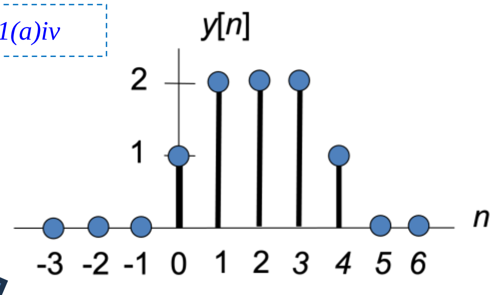
Problem 2.2 *Convolution*. 24 points.

(a) Compute and plot $y[n] = h[n] * x[n]$ using the discrete-time rectangular pulses below. *12 points*.

HW 7.1(a)iv







$$\boldsymbol{y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}$$

$$\boldsymbol{y[n] = h[0]\, x[n] + h[1]\, x[n-1] = x[n] + x[n-1]}$$

Alternate solution using MATLAB:

```
h = [1 1];
x = [1 1 1 1];
y = conv(h, x);    % [1 2 2 2 1]
```

(b) Compute and plot $y[n] = h[n] * x[n]$ using the discrete-time rectangular pulses below. *12 points*.

$$h[n] = \begin{cases} 1 & \text{for } 0 \le n \le L_h - 1 \\ 0 & \text{otherwise} \end{cases}$$

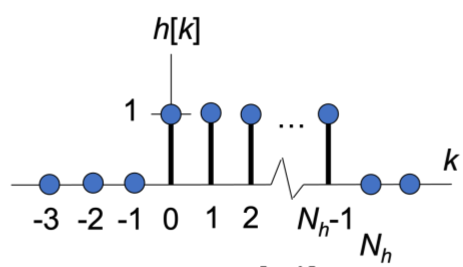
$$x[n] = \begin{cases} 1 & \text{for } 0 \le n \le L_x - 1 \\ 0 & \text{otherwise} \end{cases}$$

Same as HW 7.1(a)iii.

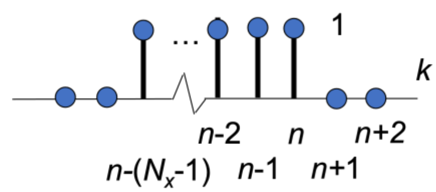
where $L_h < L_x$ and both L_h and L_x are positive integers. Give your answer in terms of L_h and L_x.

Define $L_{min} = \min(L_h, L_x)$ and $L_{max} = \max(L_h, L_x)$. Convolution result is a causal trapezoid of $L_y = L_h + L_x - 1$ samples in duration.

$$\boldsymbol{y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=0}^{L_h-1} h[k]\, x[n-k]}$$



x[n-k]



There are five cases for flip-and-slide convolution to consider:

1. ***No overlap*. $n < 0$. Amplitude is 0.**
2. ***Partial overlap*. $0 \le n \le L_{min} - 1$. Amplitude is $(n+1)$. Initial overlap of one sample at $n = 0$ with a product of one. Each shift by one in n adds one more overlapping sample with product of one.**

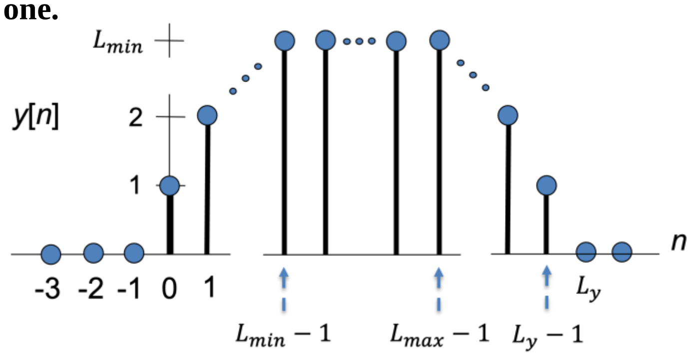
$$\boldsymbol{y[n] = \sum_{k=0}^{n} h[k]\, x[n-k] = \sum_{k=0}^{n} 1 = (n+1)}$$

3. ***Complete overlap*. $L_{min} - 1 \le n \le L_{max} - 1$. Amplitude is L_{min}. Here, L_{min} samples overlap, and each sample has a value of one.**
4. ***Partial overlap*. $L_{max} \le n \le L_y - 1$. Amplitude is $L_y - n$. Amplitude reduces by one each time n is incremented.**

$$\boldsymbol{y[n] = \sum_{k=n-(L_x-1)}^{L_h-1} 1 = (L_h - 1) + (L_x - 1) + 1 - n}$$

$$\boldsymbol{y[n] = L_h + L_x - 1 - n = L_y - n}$$

5. **No overlap. $n \ge L_y$. Amplitude is 0.**



Problem 2.3 *System Identification*. 27 points.

You are given several causal discrete-time linear time-invariant (LTI) systems each with unknown impulse response but you are able to observe the input signal $x[n]$ and output signal $y[n]$ for $-\infty < n < \infty$.

For reference, the unit step function $u[n]$ is defined as

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$y[n]$	$Y(z)$	Region of Convergence
$\delta[n]$	1	all z
$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z \rvert > 1$
$a^n\, u[n]$	$\frac{1}{1-a\,z^{-1}}$	$\lvert z \rvert > \lvert a \rvert$

(a) When input is $x[n] = \delta[n] - \delta[n-1]$, output is $y[n] = \delta[n] - 2\,\delta[n-1] + \delta[n-2]$. Find the impulse response $h[n]$. *9 points*. **This is from mini-project 2.**

Since the input signal is two samples in duration and the output signal is three samples in duration, the impulse response is two samples in duration because $y[n] = h[n] * x[n]$.

***Time-domain approach*. $y[n] = \delta[n] = u[n] - u[n-1]$. Since $x[n] = u[n]$, we can write $y[n] = x[n] - x[n-1]$ and hence $h[n] = \delta[n] - \delta[n-1]$.**

***Deconvolution approach*. Assume the LTI system is an FIR filter observed for $n \geq 0$:**

$$y[0] = h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] + \cdots h[N-1]\, x[n-(N-1)]$$

All initial conditions are zero as a necessary condition for LTI properties to hold:
$y[0] = h[0]\, x[0]$ so $1 = h[0]$ because $y[0] = 1$ and $x[0] = 1$ so $h[0] = 1$
$y[1] = h[0]\, x[1] + h[1]\, x[0]$ which is $0 = h[0] + h[1]$ so $h[1] = -1$
$y[2] = h[0]\, x[2] + h[1]\, x[1] + h[2]\, x[0]$ which is $0 = h[0] + h[1] + h[2]$ so $h[2] = 0$
We can check to see that $h[n] = \delta[n] - \delta[n-1]$ convolved with $u[n]$ is $\delta[n]$.

***Z-domain approach*. An equalizer problem in disguise. We are trying to find an LTI system $h[n]$ so that $h[n] * u[n] = \delta[n]$. In the z-domain, $H(z)\, U(z) = 1$ which means that $H(z) = \frac{1}{U(z)} = \frac{1}{\frac{1}{1-z^{-1}}} = 1 - z^{-1}$ for $z \neq 0$. Inverse z-transform is $h[n] = \delta[n] - \delta[n-1]$.**

(b) When input is $x[n] = 0.9^n\, u[n]$, output $y[n] = \delta[n]$ where $\delta[n]$ is the discrete-time impulse:

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response $h[n]$. *9 points*. **This is from mini-project 2.**

This is from mini-project 2.

***Z-domain approach*. For input $x[n] = 0.9^n\, u[n]$ and output $y[n] = \delta[n]$,**

$$X(z) = \frac{1}{1 - 0.9\, z^{-1}} \text{ for } |z| > 0.9 \text{ and } Y(z) = 1 \text{ for all } z$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\frac{1}{1-0.9\, z^{-1}}} = 1 - 0.9\, z^{-1} \text{ for } z \neq 0$$

Taking the inverse z-transform of $H(z) = 1 - 0.9\, z^{-1}$ gives $h[n] = \delta[n] - 0.9\, \delta[n-1]$.

(c) When the input is $x[n] = u[n]$, the output is $y[n]$ is a rectangular pulse of L samples in duration:

$$y[n] = \begin{cases} 1 & \text{for } 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response $h[n]$. *9 points*.

Time-domain approach. $y[n] = \delta[n] = u[n] - u[n-L]$. **Since** $x[n] = u[n]$, **we can write** $y[n] = x[n] - x[n-L]$ **and hence** $h[n] = \delta[n] - \delta[n-L]$.

Deconvolution approach. **Assume the LTI system is an FIR filter observed for** $n \geq 0$ **:**

$$y[0] = h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + \cdots h[N-1]\,x[n-(N-1)]$$

All initial conditions are zero as a necessary condition for LTI properties to hold:

$y[0] = h[0]\,x[0]$ **so** $1 = h[0]$ **because** $y[0] = 1$ **and** $x[0] = 1$ **so** $h[0] = 1$

$y[1] = h[0]\,x[1] + h[1]\,x[0]$ **which is** $1 = h[0] + h[1]$ **so** $h[1] = 0$

$y[2] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0]$ **which is** $1 = h[0] + h[1] + h[2]$ **so** $h[2] = 0$

If $h[n] = \delta[n]$, **then** $h[n] * u[n] \neq y[n]$. **So, we keep computing** $h[n]$ **values.**

$y[L] = h[0]\,x[L] + h[1]\,x[L-1] + \cdots + h[L]\,x[0]$ **which is**

$0 = h[0] + h[1] + \cdots + h[L]$ **so** $h[L] = 0$

We check to see that $h[n] = \delta[n] - \delta[n-L]$ **convolved with** $u[n]$ **is** $y[n]$.

Z-domain approach. **We're finding LTI system** $h[n]$ **so that** $h[n] * u[n]$ **is rectangular pulse of** L **samples in duration. In the** z**-domain,** $H(z)\,U(z) = 1 + z^{-1} + \cdots + z^{-(L-1)}$ **which means**

$$H(z) = \frac{1 + z^{-1} + \cdots + z^{-(L-1)}}{\frac{1}{1 - z^{-1}}} = \left(1 + z^{-1} + \cdots + z^{-(L-1)}\right)\left(1 - z^{-1}\right) = 1 - z^{-L} \text{ for } z \neq 0$$

Taking the inverse z-transform gives $h[n] = \delta[n] - \delta[n-L]$. **In Matlab, polynomial multiplication is computed using the conv command, e.g. conv([1 1 1 1 1 1 1], [1 -1]).**

```
%% Deconvolution by Prof. Brian L. Evans.
%% Keep in mind the first element in a
%% MATLAB vector has index 1 and not 0.

%% USAGE
%% FIR filters convolve the input signal
%% and the FIR filter impulse response
%% (which is equal to the filter coeffs).
%% When input signal has finite length,
%% the output is finite length:
%%
%% LengthOfy = LengthOfx + NumCoeffs - 1
%%
%% Given finite-length signals x and y,
%% we can determine how many filter
%% coefficients b there are.
%%
%% If the input signal is infinite in
%% length, then the output could be
%% either infinite or finite in length.

%% Define input and output signals. Give
%% an equal number of x and y values if
%% x is to be considered infinite length.
```

```
x = [ 1  -1 ];        %% Midterm 2.3(a)
y = [ 1  -2  1 ];

%% Determine Nmax based on input signal
%% Finite-length   length(y) - length(x) + 1
%% Infinite-length length(x)
if ( length(x) == length(y)  )
    Nmax = length(x);
else
    Nmax = length(y) - length(x) + 1;
end

b = zeros(1, Nmax);
b(1) = y(1) / x(1);
for k = 2:Nmax
    numer = y(k);
    n = k;
    for m = 1:(k-1)
        if (n >= 1)
            numer = numer - b(m) * x(n);
        end
        n = n - 1;
    end
    b(k) = numer / x(1);
end
```

SPFirst Ch. 6, 7 & 8	Lecture Slides 10-9 to 10-11 and 11-6 to 11-11
Midterm 2: F17 2.3 & 2.4 and F18 2.3 and F21 2.3	HW 5.4 & 7.2

Problem 2.4. *Filter Design.* 22 points.

Consider designing discrete-time linear time-invariant (LTI) infinite impulse response (IIR) filters.

In this problem, <u>all the poles and zeros will be real-valued</u>.

In each part below, design a biquad by placing real-valued poles and zeros to achieve the indicated frequency selectivity (lowpass, highpass, bandpass, bandstop, allpass or notch) or indicate that no such biquad with real-valued poles and zeros could be designed.

Please use O to indicate real-valued zero locations and X to indicate real-valued pole locations.

With poles inside unit circle, we convert transfer function *H*(*z*) into the discrete-time frequency domain by substituting *z* = exp(*j* ω). We take the absolute value to get the magnitude response:

$$H(z) = C\frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)} \; ; \; |H(e^{j\omega})| = \left| C\frac{(e^{j\omega} - z_0)(e^{j\omega} - z_1)}{(e^{j\omega} - p_0)(e^{j\omega} - p_1)} \right| = |C| \frac{|e^{j\omega} - z_0||e^{j\omega} - z_1|}{|e^{j\omega} - p_0||e^{j\omega} - p_1|}$$

Frequency (angle) of a pole near but inside unit circle indicates a peak in magnitude response at that frequency. From Euclidean distance $|e^{j\omega} - p_0|$ in the denominator, the minimum distance occurs when ω is equal to the angle of the pole p_0. Frequency (angle) of a zero on/near the unit circle indicate indicates a frequency that will be zeroed out/greatly attenuated.

(a) ***A first-order LTI IIR filter*** has zero z_0 and pole p_0; its transfer function is $H(z) = C\frac{(z - z_0)}{(z - p_0)}$ where C is a constant. Give numeric values for zero z_0 and pole p_0 to give each magnitude response below, place the zero and pole on the pole-zero diagram, and explain your reasoning. *10 points.*

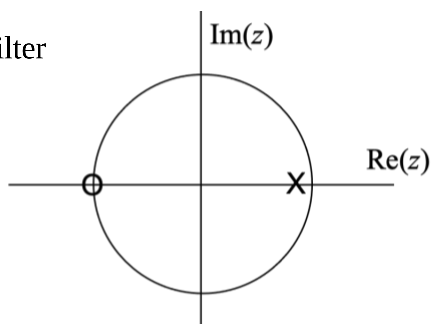
(1) Lowpass filter

Pole

$p_0 = 0.9$

Zero

$z_0 = -1$



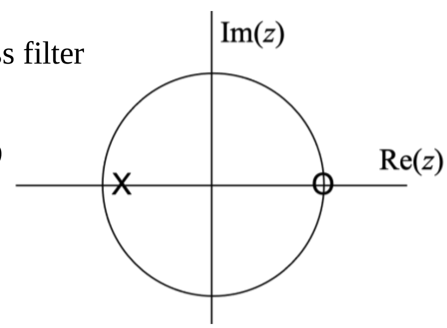
(2) Highpass filter

Pole

$p_0 = -0.9$

Zero

$z_0 = 1$



(b) ***A second-order LTI IIR filter*** has zeros z_0 and z_1 and poles p_0 and p_1, and its transfer function in is $H(z) = C\frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)}$ where C is a constant. Give numeric values for zeros z_0 and z_1 and poles p_0 and p_1 to give each magnitude response below, place the zeros and poles on the pole-zero diagram, and explain your reasoning. *12 points.*

(3) Bandpass filter

Poles

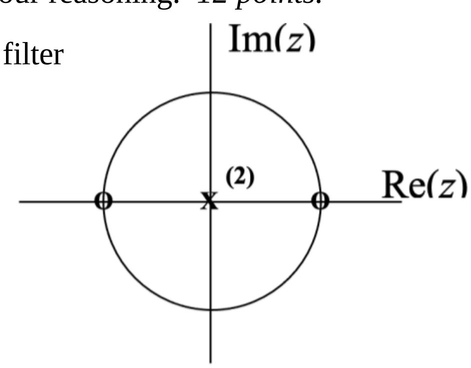
$p_0 = 0$

$p_1 = 0$

Zeros

$z_0 = -1$

$z_1 = 1$



(4) Bandstop filter

Poles

$p_0 = -0.9$

$p_1 = 0.9$

Zeros

$z_0 = 0$

$z_1 = 0$